The Inverse Laplace Transform

1. If $\mathcal{L}\{f(t)\} = F(s)$, then the *inverse Laplace transform* of $F(s)$ is

$$\mathcal{L}^{-1}\{F(s)\} = f(t). \tag{1}$$

The inverse transform $\mathcal{L}^{-1}$ is a linear operator:

$$\mathcal{L}^{-1}\{F(s) + G(s)\} = \mathcal{L}^{-1}\{F(s)\} + \mathcal{L}^{-1}\{G(s)\}, \tag{2}$$

and

$$\mathcal{L}^{-1}\{cF(s)\} = c\mathcal{L}^{-1}\{F(s)\}, \tag{3}$$

for any constant c.

2. **Example:** The inverse Laplace transform of

$$U(s) = \frac{1}{s^3} + \frac{6}{s^2+4},$$

is

$$\begin{aligned} u(t) &= \mathcal{L}^{-1}\{U(s)\} \\ &= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} + 3\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} \\ &= \frac{s^2}{2} + 3\sin 2t. \end{aligned} \tag{4}$$

3. **Example:** Suppose you want to find the inverse Laplace transform $x(t)$ of

$$X(s) = \frac{1}{(s+1)^4} + \frac{s-3}{(s-3)^2+6}.$$

Just use the shift property (paragraph 11 from the previous set of notes):

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^4}\right\} + \mathcal{L}^{-1}\left\{\frac{s-3}{(s-3)^2+6}\right\} \\ &= \frac{e^{-t}\, t^3}{6} + e^{3t}\cos\sqrt{6}t. \end{aligned}$$

4. **Example:** Let $y(t)$ be the inverse Laplace transform of

$$Y(s) = \frac{e^{-3s}\, s}{s^2+4}.$$

Don't worry about the exponential term. Since the inverse transform of $s/(s^2+4)$ is $\cos 2t$, we have by the switchig property (paragraph 12 from the previous notes):

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{e^{-3s}\, s}{s^2+4}\right\} \\ &= H(t-3)\cos 2(t-3). \end{aligned}$$

5. **Example:** Let $G(s) = s(s^2+4s+5)^{-1}$. The inverse transform of $G(s)$ is

$$\begin{aligned} g(t) &= \mathcal{L}^{-1}\left\{\frac{s}{s^2+4s+5}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{s}{(s+2)^2+1}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{2}{(s+2)^2+1}\right\} \\ &= e^{-2t}\cos t - 2e^{-2t}\sin t. \end{aligned} \tag{5}$$

6. There is usually more than one way to invert the Laplace transform. For example, let $F(s) = (s^2+4s)^{-1}$. You could compute the inverse transform of this function by completing the square:

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\left\{\frac{1}{s^2+4s}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2-4}\right\} \\ &= \frac{1}{2}\,\mathcal{L}^{-1}\left\{\frac{2}{(s+2)^2-4}\right\} \\ &= \frac{1}{2}\,e^{-2t}\sinh 2t. \end{aligned} \tag{6}$$

You could also use the partial fraction decomposition (PFD) of $F(s)$:

$$F(s) = \frac{1}{s(s+4)} = \frac{1}{4s} - \frac{1}{4(s+4)}.$$

Therefore,

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\left\{F(s)\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{4s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{4(s+4)}\right\} \\ &= \frac{1}{4} - \frac{1}{4}e^{-4t} \\ &= \frac{1}{2}\,e^{-2t}\sinh 2t. \end{aligned} \tag{7}$$

7. Example: Compute the inverse Laplace transform $q(t)$ of

$$Q(s) = \frac{3s}{(s^2+1)^2}.$$

You could compute $q(t)$ by partial fractions, but there's a less tedious way. Note that

$$Q(s) = -\frac{3}{2}\frac{d}{ds}\frac{1}{s^2+1}.$$

Hence,

$$\begin{aligned} q(t) &= \mathcal{L}^{-1}\{Q(s)\} \\ &= -\frac{3}{2}\mathcal{L}^{-1}\left\{\frac{d}{ds}\frac{1}{s^2+1}\right\} \\ &= \frac{3}{2}t\sin t. \end{aligned} \tag{8}$$

8. Definition: The *convolution* of functions $f(t)$ and $g(t)$ is

$$(f*g)(t) = \int_0^t f(t-v)g(v)\,dv. \tag{9}$$

As we showed in class, the convolution is commutative:

$$(f*g)(t) = \int_0^t f(t-v)g(v)\,dv = \int_0^t g(t-v)f(v)\,dv = (g*f)(t). \tag{10}$$

9. Example: Let $f(t) = t$ and $g(t) = e^t$. The convolution of f and g is

$$\begin{aligned} (f*g)(t) &= \int_0^t (t-v)e^v\,dv \\ &= t\int_0^t e^v\,dv - \int_0^t ve^v\,dv \\ &= e^t - t - 1. \end{aligned} \tag{11}$$

10. Proposition: (The Convolution Theorem) If the Laplace transforms of $f(t)$ and $g(t)$ are $F(s)$ and $G(s)$ respectively, then

$$\mathcal{L}\{(f*g)(t)\} = F(s)G(s), \tag{12}$$

that is,

$$\mathcal{L}^{-1}\left\{F(s)G(s)\right\} = (f * g)(t). \tag{13}$$

11. Suppose that you want to find the inverse transform $x(t)$ of $X(s)$. If you can write $X(s)$ as a product $F(s)G(s)$ where $f(t)$ and $g(t)$ are known, then by the above result, $x(t) = (f * g)(t)$.

12. **Example:** Consider the previous example: Find the inverse transform $q(s)$ of

$$Q(s) = \frac{3s}{(s^2+1)^2}.$$

Write $Q(s) = F(s)G(s)$, where

$$F(s) = \frac{3}{s^2+1},$$

and

$$G(s) = \frac{s}{s^2+1}.$$

The inverse transforms are of $F(s)$ and $G(s)$ are $f(t) = 3\sin t$ and $g(t) = \cos t$. Therefore

$$\begin{aligned} q(s) &= \mathcal{L}^{-1}\left\{Q(s)\right\} \\ &= \mathcal{L}^{-1}\left\{F(s)G(s)\right\} \\ &= (f * g)(t) \\ &= 3\int_0^t \sin(t-v)\cos v\, dv. \end{aligned} \tag{14}$$

Even if you stop here, you at least have a fairly simple, compact expression for $q(s)$. To do the integral (14), use the trigonometric identity

$$\sin A \cos B = \frac{\sin(A+B) + \sin(A-B)}{2}.$$

With this, (14) becomes

$$\begin{aligned} q(s) &= \frac{3}{2}\int_0^t \sin t\, dv + \int_0^t \sin(t-2v)\, dv \\ &= \frac{3}{2} t \sin t. \end{aligned} \tag{15}$$

13. **Example:** Find the inverse Laplace transform $x(t)$ of the function

$$X(s) = \frac{1}{s(s^2+4)}.$$

If you want to use the convolution theorem, write $X(s)$ as a product:

$$X(s) = \frac{1}{s}\frac{1}{s^2+4}.$$

Since

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1,$$

and

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\sin 2t,$$

we have

$$\begin{aligned} x(t) &= \frac{1}{2}\int_0^t \sin 2v\, dv \\ &= \frac{1}{4}(1-\cos 2t). \end{aligned}$$

You could also use the PFD:

$$X(s) = \frac{1}{4s} - \frac{s}{4(s^2+4)}.$$

Therefore,

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}\left\{\frac{1}{4s}\right\} - \mathcal{L}^{-1}\left\{\frac{s}{4(s^2+4)}\right\} \\ &= \frac{1}{4}(1-\cos 2t). \end{aligned}$$